International Journal of Theoretical Physics, Vol. 42, No. 10, October 2003 (© 2003)

Gamow Vectors in Exactly Solvable Models

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Received June 30, 2003

Through two examples: the Friedrichs model and a particular case of central potential scattering, we illustrate the way of constructing Gamow vectors.

KEY WORDS: resonances; Gamow vectors; inverse scattering.

1. INTRODUCTION

Gamow vectors have been introduced in order to define a vector state for exponentially decaying resonances in Quantum Mechanics and named after the pioneering work by Gamow in 1928 (Gamow, 1928). Although Gamow used a simple model for the alpha decay, he obtained properties for his Gamow vectors that are general. In particular, the radial part of their wave functions goes to infinity as $r \mapsto \infty$. Bound state eigenfunctions and Gamow state eigenfunctions are solutions of the same radial differential equation with similar boundary conditions. However, while bound state eigenfunctions are square integrable functions and the corresponding eigenvalues are real, Gamow state eigenfunctions are not square integrable and the corresponding eigenvalues are complex. Making abstraction of the name of the eigenfunction, we may refer to the expression on the left hand side of the radial differential equation as "formal Hamiltonian," which may be considered as an operator defined on a space of functions. If this space is the Hilbert space of square integrable functions which vanish at the origin (the space $L^2(\mathbb{R}^+)$), the Hamiltonian is self-adjoint and has only real eigenvalues. The Gamow eigenfunctions are not elements of this space. If we want to consider also the Gamow eigenfunctions as elements of a space, one has to consider the same "formal

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0020-7748/03/1000-2389/0 © 2003 Plenum Publishing Corporation

Hamiltonian" as an operator acting on a larger space, one which contains the Hilbert space of square integrable functions and the Gamow eigenfunctions. One such space is the rigged Hilbert space, where the "formal Hamiltonian" can have eigenfunctions which are not square integrable and have complex energy eigenvalues. On the Hilbert subspace of the rigged Hilbert space, the formal Hamiltonian is selfadjoint.

Gamow also considered that resonances are produced in resonance scattering. This implies the existence of two Hamiltonians: the unperturbed or free Hamiltonian H_0 and the total Hamiltonian H. Responsible for the resonance process is the presence of a potential $V = H - H_0$. These ideas appear to be very useful in the study of resonances and have been used since then on by many authors (Bohm, 1993; Brändas and Elander, 1989; Goldberger and Watson, 1964; Newton, 1982; Nussenzveig, 1972).

In this context and under rather general conditions, there are several equivalent ways to introduce resonances in Quantum Mechanics: i.) Pairs of poles in the analytic continuation of the S-matrix in the momentum or energy representation (Bohm, 1993; Newton, 1982; Nussenzveig, 1972), ii.) Poles in the analytic continuation of the reduced resolvent through the spectrum of H (Friedrichs, 1948; Horwitz and Marchand, 1971), iii.) Complex eigenvalues of an analytically dilated Hamiltonian (Reed and Simon, 1979), iv.) Complex eigenvalues of the Hamiltonian with eigenvectors satisfying "purely outgoing boundary conditions" (de la Madrid, 2000, 2001), v.) Complex eigenvalues of a dissipative generator of a semigroup in the Lax–Phillips theory (Lax and Philips, 1967; Strauss *et al.*, 2000).

Then, Gamow vectors are generalized eigenvectors of H with complex eigenvalues (Bohm, 1993; Bohm and Gadella, 1989) at the resonance points. These eigenvectors belong to non Hilbert equipments of the Hilbert space on which H acts, called rigged Hilbert spaces or Gelfand triplets (Bohm, 1993; Gadella and Gómez, 2003).

This is a brief review that intends to show how Gamow vectors are obtained. Although the procedure is very general, the two examples presented here illustrate the method. The first example is the Friedrichs model in which we allow the resonance poles to have multiplicity one or two. In the second model, we assume a spherically symmetric potential to work with a fixed value of the orbital angular momentum, in our case $l = 0$, and then construct the S matrix, in the momentum representation, with poles at given points.

2. GAMOW VECTORS IN THE FRIEDRICHS MODEL

2.1. The Friedrichs Model in Hilbert Space

In this paper, we shall deal with the simplest form of the Friedrichs model (Antoniou and Prigogine, 1993; Exner, 1985; Friedrichs, 1948; Horwitz and

Marchand, 1971). In this version of the Friedrichs model, the free Hamiltonian H_0 has a simple continuous spectrum, which is $\mathbb{R}^+ \equiv [0, \infty)$, plus an eigenvalue ω_0 embedded in this continuous spectrum ($\omega_0 > 0$).

The Hilbert space of this system in the energy representation is the direct sum

$$\mathcal{H} := \mathbb{C} \oplus L^2(\mathbb{R}^+), \tag{1}$$

so that, any $\psi \in \mathcal{H}$ can be represented as

$$\psi = \begin{pmatrix} \alpha \\ \varphi(\omega) \end{pmatrix}, \tag{2}$$

where α is a complex number. The function $\varphi(\omega)$ is square integrable on the interval $[0, \infty)$, i.e., $\varphi(\omega) \in L^2(\mathbb{R}^+)$, where $\mathbb{R}^+ := [0, \infty)$. The scalar product of two vectors in $\mathcal{H}$ is given by

$$\left(\begin{pmatrix} \alpha \\ \varphi(\omega) \end{pmatrix}, \begin{pmatrix} \beta \\ \eta(\omega) \end{pmatrix} \right) = \alpha^* \beta + \int_0^\infty \varphi^*(\omega) \eta(\omega)\, d\omega. \tag{3}$$

Here, β is a complex number and $\eta(\omega) \in L^2(\mathbb{R}^+)$. The domain of H_0 (the subspace of all vectors $\psi \in \mathcal{H}$ such that $H_0 \psi \in \mathcal{H}$) is given by the vectors in $\mathcal{H}$ of the form (2) such that $\omega\varphi(\omega) \in L^2(\mathbb{R}^+)$. The action of H_0 in (2) is given by

$$H_0 \begin{pmatrix} \alpha \\ \varphi(\omega) \end{pmatrix} = \begin{pmatrix} \omega_0\, \alpha \\ \omega\, \varphi(\omega) \end{pmatrix}. \tag{4}$$

Observe that the vector

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5}$$

is an eigenvector of H_0 with eigenvalue ω_0, i.e., $H_0|1\rangle = \omega_0|1\rangle$. As

$$H_0 \begin{pmatrix} 0 \\ \varphi(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ \omega\, \varphi(\omega) \end{pmatrix}, \tag{6}$$

i.e., H_0 is the multiplication operator for the second component on the interval $\mathbb{R}^+ = [0, \infty)$, we conclude that H_0 has a non degenerate absolutely continuous spectrum that coincides with $[0, \infty)$ plus an eigenvalue, ω_0. Since $\omega_0 > 0$ by construction, the eigenvalue of H_0 is embedded in its continuous spectrum.

The potential V is defined as an interaction between the discrete and continuous parts of H_0. Its form is the following:

$$V\psi = \begin{pmatrix} \int_0^\infty f(\omega)\, \varphi(\omega)\, d\omega \\ \alpha f^*(\omega) \end{pmatrix}, \tag{7}$$

where ψ is the arbitrary vector given in (2) and $f(\omega)$ is a function on $\mathbb{R}^+$ called the form factor. Observe that if we want $V\psi$ to be in $\mathcal{H}$, the form factor $f(\omega)$ must be in $L^2(\mathbb{R}^+)$. The total Hamiltonian is given by $H := H_0 + \lambda V$, where λ is a real parameter. This Hamiltonian has simple continuous spectrum only, without discrete eigenvalues, which is $\mathbb{R}^+ = [0, \infty)$.

What has happened with the eigenvalue of H_0? As the interaction is switched on, this eigenvalue is *dissolved* in the continuous spectrum of H_0. Nevertheless, the corresponding bound state *reappears* as a resonance as we shall explain next.

In order to study the resonances in the Friedrichs model, it is customary to define the *reduced resolvent* of H, which is the projection to the subspace spanned by $|1\rangle$ of the resolvent of H:

$$|1\rangle\langle 1|\frac{1}{z-H}|1\rangle\langle 1| = \frac{1}{\eta(z)}|1\rangle\langle 1| \tag{8}$$

with (Exner, 1985)

$$\eta(z) = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega}\, d\omega. \tag{9}$$

It has been shown that, under certain conditions (Antoniou *et al.*, 1998; Exner, 1985) imposed to $|f(\omega)|^2$, the function $\eta(z)$ defined in (9) is complex analytic with no singularities on the complex plane other than a branch cut coinciding with the positive semiaxis $\mathbb{R}^+$ provided that $\eta(0) > 0$. It admits respective analytic continuations through the cut from above to below (from the upper to the lower half plane) $\eta_+(z)$ and from below to above (from the lower to the upper half plane) $\eta_-(z)$. The continuation $\eta_+(z)$ has a zero at z_0 with $\mathfrak{I}\{z_0\} < 0$, which is an analytic function on the coupling parameter λ on a neighborhood of zero. Analogously, $\eta_-(z)$ admits a zero at z_0^*, which is also analytic function of λ.

The zeroes z_0 and z_0^* are therefore the poles of the reduced resolvent. We define the *resonance poles* of the pair (H_0, H) as the zeroes, of the analytic continuations through the spectrum, of $\eta_+(z)$ and $\eta_-(z)$.

It is possible to show that the scattering Møller wave operators are well defined in the Friedrichs model (Exner, 1985). Therefore, the S operator exists. Since the continuous spectrum of the Friedrichs model is not degenerated, the form of the S operator is

$$S\begin{pmatrix}\alpha\\ \varphi(\omega)\end{pmatrix} = \begin{pmatrix}\alpha\\ S(\omega)\,\varphi(\omega)\end{pmatrix}.$$

The functions $S(\omega)$ is analytic with a branch cut in the positive semiaxis $\mathbb{R}^+$. It admits analytic continuations from above to below and from below to above with respective poles at the points z_0 and z_0^*, exactly the poles of the reduced resolvent (Antoniou and Melnikov, 1998; Horwitz and Marchand, 1971). It is therefore, legitimate to look at z_0 and z_0^* as resonance poles for the pair (H_0, H).

The condition that $|f(\omega)|^2$ should be the restriction to $\mathbb{R}^+$ of an entire analytic function can be weakened. For instance in (Antoniou *et al.*, 1998), we have constructed a Friedrichs model for which $|f(\omega)|^2$ is analytic on the complex variable ω with a branch cut in the positive semiaxis.

We want to recall that the zeroes of the analytic functions $\eta_+(z)$ and $\eta_-(z)$ are, in general, analytic functions of the coupling parameter λ (Exner, 1985). This means that z_0 as well as z_0^* go to ω_0 as $\lambda \mapsto 0$. We see that as the interaction λV is switched on, the bound state is transformed into a resonance (under a simple condition such as $\eta(0) > 0$).

2.2. The Friedrichs Model in Rigged Hilbert Space

According to the Gelfand–Maurin spectral theorem, there exists a rigged Hilbert space[6] $\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^\times$ such that $H_0\mathbf{\Phi} \subset \mathbf{\Phi}$, H_0 is continuous on $\mathbf{\Phi}$ and for any $\omega \in \mathbb{R}^+$, the absolutely continuous spectrum of H_0, there exists a $|\omega\rangle \in \mathbf{\Phi}^\times$ with $H_0|\omega\rangle = \omega|\omega\rangle$. As H_0 still has the eigenvector $|1\rangle$, the spectral decomposition of H_0 is then,

$$H_0 = \omega_0|1\rangle\langle 1| + \int_0^\infty \omega|\omega\rangle\langle\omega|\, d\omega. \tag{10}$$

Then, V and $\psi \in \mathcal{H}$ must have respectively the following form:

$$V = \int_0^\infty (f^*(\omega)|\omega\rangle\langle 1| + f(\omega)|1\rangle\langle\omega|)\, d\omega, \tag{11}$$

$$\psi = \alpha|1\rangle + \int_0^\infty \varphi(\omega)|\omega\rangle\, d\omega. \tag{12}$$

With the new notation, the action of V on ψ can be obtained easily if we note that

$$\langle 1|1\rangle = 1$$

$$\langle 1|\omega\rangle = \langle\omega|1\rangle = 0$$

[6] A rigged Hilbert space is a triplet of spaces $\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^\times$, where $\mathcal{H}$ is an infinite dimensional separable Hilbert space, $\mathbf{\Phi}$ a dense subspace of $\mathcal{H}$ endowed with a topology with more open sets (and therefore, less Cauchy sequences) than the topology on $\mathcal{H}$ and $\mathbf{\Phi}^\times$ is the antidual of $\mathbf{\Phi}$. Being given a topological vector space $\mathbf{\Phi}$, its antidual, $\mathbf{\Phi}^\times$, is the vector space whose elements are continuous antilinear mappings from $\mathbf{\Phi}$ into $\mathbb{C}$. The property of antilinearity is characterized as follows: If $\varphi, \psi \in \mathbf{\Phi}$, $\alpha, \beta \in \mathbb{C}$, and $F \in \mathbf{\Phi}^\times$, we have that

$$F(\alpha\varphi + \beta\psi) = \alpha^* F(\varphi) + \beta^* F(\psi),$$

where the star denotes complex conjugation. For more details on rigged Hilbert spaces, see for instance (Antoine, 1969, 1998; Bohm and Gadella, 1989) and references therein. See also (Gadella and Gómez, 2003) in this volume.

$$\langle\omega|\omega'\rangle = \langle\omega'|\omega\rangle = \delta(\omega - \omega'). \tag{13}$$

Choice of Φ. We have two possibilities: Let S be the Schwartz space[7] and let $\mathcal{H}^2_\pm$ be the spaces of Hardy functions on the upper (+) or lower (−) half planes (Bohm and Gadella, 1989; Koosis, 1980, 1990), where $S \cap \mathcal{H}^2_\pm|_{\mathbb{R}^+}$ is the space of the restrictions to the positive semiaxis $\mathbb{R}^+$ of the functions in $S \cap \mathcal{H}^2_\pm$. Then,

$$\mathbf{\Phi}_\pm := \mathbb{C} \oplus \left(S \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}\right),$$

so that

$$\mathbb{C} \oplus \left(S \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}\right) \subset \mathbb{C} \oplus L^2(\mathbb{R}^+) \subset \mathbb{C} \oplus \left(S \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}\right)^\times$$

is a RHS, where

$$\mathbf{\Phi}^\times_\pm := \mathbb{C} \oplus \left(S \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}\right)^\times .$$

Now, if the form factor $f(\omega) \in L^2(\mathbb{R}^+)$ or if $f(\omega)$ is a polynomial in the variable ω or even a Dirac delta, $\delta(\omega)$, then the total Hamiltonian H is a continuous operator from $\mathbf{\Phi}_\pm$ into $\mathbf{\Phi}^\times_\pm$. In some cases the domain of H can be enlarged to include the Gamow vectors.

2.3. Gamow Vectors for the Friedrichs Model

The Gamow vectors (Antoniou and Prigogine, 1993; Bohm, 1993; Bohm and Gadella, 1989), $|f_0\rangle$ and $|\tilde{f}_0\rangle$, are characterized by the following property:

$$H|f_0\rangle = z_0|f_0\rangle; \quad H|\tilde{f}_0\rangle = z_0^*|\tilde{f}_0\rangle, \tag{14}$$

since z_0 and z_0^* are complex numbers, $|f_0\rangle$ and $|\tilde{f}_0\rangle$ are not normalizable vectors. As a matter of fact, $|f_0\rangle \in \mathbf{\Phi}^\times_+$ and $|\tilde{f}_0\rangle \in \mathbf{\Phi}^\times_-$.

We shall see how to obtain the Gamow vectors corresponding to the resonance states of the Friedrichs model. First, we shall assume that the zeroes of $\eta_\pm(z)$ at z_0 and z_0^* are simple. Then, to obtain the Gamow vector, consider the following eigenvalue equation valid for any $x > 0$:

$$(H - x)\Psi(x) = 0, \tag{15}$$

where $\Psi(x)$ is the eigenvector of H for the eigenvalue x. As the eigenvectors $|1\rangle$ and $|\omega\rangle$ form a complete system, we have

$$\Psi(x) = \psi(x)|1\rangle + \int_0^\infty \psi(x,\omega)|\omega\rangle\, d\omega. \tag{16}$$

[7] Space of indefinitely differentiable complex continuous functions on the real line, that they and their derivatives go to zero at the infinite faster than the inverse of a polynomial.

Now, if we carry (16) into (15), we obtain the following system of integral equations:

$$(\omega_0 - x)\psi(\omega) + \lambda \int_0^\infty \psi(x,\omega) f^*(\omega)\, d\omega = 0 \tag{17}$$

$$(\omega - x)\psi(x,\omega) + \lambda f(\omega)\psi(\omega) = 0. \tag{18}$$

To solve this system, we write $\psi(\omega)$ in terms of $\psi(x,\omega)$ using (18) and carry the result to (17). We obtain an integral equation, which gives as a solution:

$$\Psi_\pm(x) = |x\rangle + \lambda f^*(x) \frac{1}{\eta_+(x)} \left\{ |1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{x - \omega + i0} |\omega\rangle\, d\omega \right\}. \tag{19}$$

This is a functional in $\Phi_+^\times$. When applied to a vector in Φ_+, it gives an analytic function on the lower half plane. We say that $\Psi_+(x)$ admits analytic continuation to the lower half plane (in a weak sense). This continuation has a simple pole at z_0 so that we can write on a neighborhood of z_0:

$$\Psi_+(z) = \frac{C}{z - z_0} + o(z), \tag{20}$$

where $o(z)$ denotes a Taylor series on z, which converges on a neighborhood of z_0. From (15) and (20), we get:

$$0 = (H - z)\Psi_+(z) = \frac{1}{z - z_0}(H - z)C + (H - z)o(z), \tag{21}$$

which gives

$$(H - z_0)C = 0 \Longrightarrow HC = z_0 C. \tag{22}$$

This shows that the residue C in (20) is precisely the Gamow vector we are looking for. Thus, to obtain it, we just proceed as we do in a standard calculus of residua. In a neighborhood of z_0, we have:

$$\Psi_+(z) \approx \frac{\text{constant}}{(z - z_0)} \left\{ |1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z - \omega + i0} |\omega\rangle\, d\omega \right\} + o(z). \tag{23}$$

Since

$$\frac{1}{z - \omega + i0} = \frac{1}{z_0 - \omega + i0} - \frac{z - z_0}{(z_0 - \omega + i0)^2} + o(z), \tag{24}$$

we have that

$$\Psi_+(z) \approx \frac{\text{constant}}{(z - z_0)} \left\{ |1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z_0 - \omega + i0} |\omega\rangle\, d\omega \right\} + o(z). \tag{25}$$

Therefore, save for an irrelevant constant, we conclude that

$$C \equiv |f_0\rangle = |1\rangle + \int_0^\infty \frac{\lambda f(\omega)}{z_0 - \omega + i0} |\omega\rangle\, d\omega. \tag{26}$$

Analogously,

$$|\tilde{f}_0\rangle = |1\rangle + \int_0^\infty \frac{\lambda f^*(\omega)}{z_0^* - \omega + i0}|\omega\rangle\, d\omega. \tag{27}$$

Therefore, formulas (26) and (27) give us the two Gamow vectors for simple pole resonances at z_0 and z_0^* in the Friedrichs model. Note that $|f_0\rangle \in \mathbf{\Phi}_+^\times$ and $|\tilde{f}_0\rangle \in \mathbf{\Phi}_-^\times$.

2.4. Double Pole Resonances

The existence and the properties of the zeros of $\eta_+(z)$ and $\eta_-(z)$ depend on the form factor $f(\omega)$. We are interested here in the possibility of having Friedrichs models with degenerate resonances, i.e., with resonances produced by multiple zeros of $\eta_+(z)$ and $\eta_-(z)$. Of course, the simplest case occurs when these zeroes are double. We shall discuss here this particular example. To this end, let us choose the following form factor:

$$|f(\omega)|^2 = \frac{\sqrt{\omega}}{P(\omega)} \tag{28}$$

with

$$P(\omega) := (\omega - \alpha)(\omega - \alpha^*). \tag{29}$$

Then,

$$\eta(z) = \omega_0 - z - \pi\lambda^2 \left\{ \frac{\sqrt{-z}}{P(z)} - \frac{1}{\alpha - \alpha^*}\left(\frac{\sqrt{-\alpha}}{z - \alpha} - \frac{\sqrt{-\alpha^*}}{z - \alpha^*} \right) \right\}. \tag{30}$$

If we make the change of variables:

$$z = p^2; \quad \alpha = b^2 \tag{31}$$

and write $\varphi(p) = \eta(p^2) = \eta(z)$, we have that

$$\varphi(p) = \omega_0 - p^2 + \frac{i\pi\lambda^2}{(b - b^*)(p + b)(p - b^*)}. \tag{32}$$

If $\varphi(p)$ has a double zero at p_R, we must have:

$$\varphi(p_R) = 0; \quad \varphi'(p_R) = 0; \quad \varphi''(p_R) \neq 0, \tag{33}$$

which give four equations for six parameters. These are: ω_0, λ, $\Re p_R$, $\Im p_R$, $\Re b$, and $\Im b$. We have two free parameters that we choose as ω_0 and λ. Then,

$$b = \omega_0^{1/2} + 2i\left(\frac{\pi\lambda^2}{16\omega_0}\right)^{1/3}$$

$$p_R = \left[\omega_0 - \left(\frac{\pi\lambda^2}{16\omega_0}\right)^{2/3}\right] - \left(\frac{\pi\lambda^2}{16\omega_0}\right)^{1/3}. \tag{34}$$

Consequently,

$$\varphi(p) = -\frac{(p - p_R)^2(p + p_R^*)^2}{(p + t)(p - t^*)} \tag{35}$$

and $\varphi(p)$ has two double poles and hence $\eta(z)$. To obtain the Gamow vectors, we proceed as before and note that on a neighborhood of $z_0 = p_R^2$, we have

$$\Psi_+(z) = \frac{C_1}{(z - z_0)^2} + \frac{C_2}{(z - z_0)} + o(z). \tag{36}$$

As in the previous case, we have on a neighborhood of z_0

$$\begin{aligned}\Psi_+(z) &\approx \frac{\text{constant}}{(z - z_0)^2}\left\{|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{z - \omega + i0}|\omega\rangle\, d\omega\right\} \\ &= \frac{\text{constant}}{(z - z_0)^2}\left\{|1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{z_0 - \omega + i0}|\omega\rangle\, d\omega\right. \\ &\qquad \left. -\ \lambda(z - z_0)\int_0^\infty \frac{f(\omega)}{(z_0 - \omega + i0)^2}|\omega\rangle\, d\omega\right\}.\end{aligned} \tag{37}$$

Then,

$$C_1 = |1\rangle + \lambda\int_0^\infty \frac{f(\omega)}{z_0 - \omega + i0}|\omega\rangle\, d\omega \tag{38}$$

$$C_2 = -\lambda\int_0^\infty \frac{f(\omega)}{(z_0 - \omega + i0)^2}|\omega\rangle\, d\omega \tag{39}$$

In addition, we can prove that (Antoniou *et al.*, 1998)

$$HC_1 = z_0 C_1; \quad HC_2 = z_0 C_2 + C_1 \tag{40}$$

On the subspace of $\Phi^\times$ spanned by C_1 and C_2, the total Hamiltonian H has the following block diagonal form:

$$H = \begin{pmatrix} z_0 & 1 \\ 0 & z_0 \end{pmatrix} \tag{41}$$

This block diagonal form has been found ealier in different contexts (Brändas, 1995; Brändas and Chatzidimitriou-Dreissmann, 1989; Hernández and Mondragón, 1994; Mondragón and Hernández, 1993, 1996).

3. GAMOW VECTORS IN A CENTRAL POTENTIAL SCATTERING

If the potential is spherically symmetric, there is a procedure due to Bargmann (1949) and Theis (1956) to add a finite number of resonances at selected points, for a fixed value of the angular momentum. We want to illustrate this procedure with a simple example. Let us begin with the Hamiltonian of a free particle of mass m and spin zero. Here, there is no scattering and hence, for each value of the orbital angular momentum, the component of the S-matrix is the unity: $S_l(k) \equiv I$. We shall use here the momentum representation. Then, in the k-plane, we consider the following rational function

$$R(k) := \frac{(k-\gamma_1)(k-\gamma_2)}{(k-\beta_1)(k-\beta_2)} \tag{42}$$

where

$$\gamma_1 := a - ib, \quad \gamma_2 := -a - ib, \quad \beta_1 := c - id, \quad \beta_2 := -c - id \tag{43}$$

where a, b, c, and d are positive numbers. We look for a potential such that, the value of the S-matrix for $l = 0$ is

$$S_0(k) = \frac{R(-k)}{R(k)} = \frac{(k+\gamma_1)(k+\gamma_2)}{(k+\beta_1)(k+\beta_2)} \frac{(k-\beta_1)(k-\beta_2)}{(k-\gamma_1)(k-\gamma_2)} \tag{44}$$

and the other components of the S-matrix for $l \neq 0$, $S_l(k)$, remain unchanged and equal to one. For $l = 0$, let us consider the following solutions of the free radial Schrödinger equation:

$$\varphi(k, r) = \frac{\sin kr}{k} \quad \text{and} \quad f(k, r) = e^{ikr}. \tag{45}$$

With the help of these solutions, let us construct the following expressions:

$$x_{\beta_i}(k, r) := \frac{W[\varphi(\beta_i, r),\ f(k, r)]}{\beta_i^2 - k^2} = \frac{1}{\beta_i^2 - k^2}\left[\frac{ik}{\beta_i}\sin\beta_i r - \cos\beta_i r\right] e^{ikr} \tag{46}$$

where $i = 1, 2$ and the symbol W denotes the Wronskian of the functions between brackets. Analogusly,

$$\begin{aligned} y_{\beta_i} :&= \frac{W[\varphi(\beta_i, r), \varphi(k, r)]}{\beta_i^2 - k^2} \\ &= \frac{1}{\beta_i^2 - k^2}\left[\frac{1}{\beta_i}\sin\beta_i r\ \cos kr - \frac{1}{k}\cos\beta_i r\ \sin kr\right], \end{aligned} \tag{47}$$

and finally,

$$x_{ij}(r) := x_{\beta_j}(-\gamma_i, r) = -\frac{e^{-i\gamma_i r}}{2\beta_j}\left[\frac{e^{i\beta_j r}}{\beta_j - \gamma_i} + \frac{e^{-i\beta_j r}}{\beta_j + \gamma_i}\right]. \tag{48}$$

Now, let M be the 2×2 matrix whose entries are $x_{ij}(r)$, $i, j = 1, 2$. The desired potential is now given, in terms of the radial variable r as (Nussenzveig, 1972)

$$V(r) = -2\frac{d^2}{dr^2}\log\{\det M\} \tag{49}$$

In order to obtain the Gamow vectors, corresponding to the resonance poles $\gamma_{1,2}$, we proceed as follows (Nussenzveig, 1972): we have to obtain two functions $K_i(r)$, $i = 1, 2$ satisfying the following system of linear equations:

$$\begin{aligned} x_{11}(r)K_1(r) + x_{12}(r)K_2(r) &= e^{-i\gamma_1 r} \\ x_{21}(r)K_1(r) + x_{22}(r)K_2(r) &= e^{-i\gamma_2 r}. \end{aligned} \tag{50}$$

The solution of (50) is:

$$\begin{aligned} K_1(r) &= \frac{x_{22}(r)e^{-i\gamma_1 r} - x_{12}(r)e^{-i\gamma_2 r}}{\det M} \\ K_2(r) &= \frac{x_{11}(r)e^{-i\gamma_2 r} - x_{21}(r)e^{-i\gamma_1 r}}{\det M}. \end{aligned} \tag{51}$$

This solution is useful to construct the Gamow vectors for these two resonances. In fact, the solutions of the radial Schrödinger equation for $l = 0$ with potential given by (49) and energy $-k^2/2m$ are:

$$h(k, r) := e^{ikr} + \sum_{i=1}^{2} K_i(r)x\beta_i(k, r) \tag{52}$$

and

$$g(k, r) := \frac{\sin kr}{k} + \sum_{i=1}^{2} K_i(r)y\beta_i(k, r) \tag{53}$$

It is proven that $\det M \neq 0$. Therefore, $V(r)$ and the $K_i(r)$ are well defined. Observe that

$$\det\ M = \frac{e^{-i(\gamma_1+\gamma_2)r}}{4\beta_1\beta_2}F(r) \tag{54}$$

with

$$F(r) = Ae^{i(\beta_1+\beta_2)r} + Be^{i(\beta_1-\beta_2)r} + Ce^{i(\beta_1+\beta_2)r} + De^{-i(\beta_1-\beta_2)r} \tag{55}$$

where A, B, C, and D are constants which depend on the β_i and the γ_i. Thus, (52) and (53) are nonvanishing rational functions of exponentials.

In order to obtain the Gamow vectors, let us consider an incoming free wave function of the form e^{ikr}. To compare this incoming wave function with the outgoing wave function, we need the asymptotic form of (52) for large values of r, which is given by

$$h(k, r) \sim e^{ikr} R(-k). \tag{56}$$

The relation between the incoming wave function and the outgoing wave function is given by the S-matrix. Thus, the corresponding solution with asymptotic form $S_0(k)e^{ikr}$ is

$$\tilde{h}(k, r) = \frac{h(k, r)}{R(k)}, \tag{57}$$

The Gamow vectors should be obtained from the residues of (57) on the poles at γ_i, $i = 1, 2$. In order to prove this statement, let us consider the following equation for complex k:

$$H\psi(k, r) = \frac{k^2}{2m}\psi(k, r) \tag{58}$$

and assume that $\psi(k, r)$ is a meromorphic function of k with a simple pole at z_0. Then, we can write

$$\psi(k, r) = \frac{\phi(r)}{k - z_0} + r.t., \tag{59}$$

where r.t. in (59) stands for "regular terms." The residue $\phi(r)$ depends only on r. Then,

$$\begin{aligned} H\left\{\frac{\phi(r)}{k - z_0} + r.t.\right\} &= \frac{1}{k - z_0}H\phi(r) + H\{r.t.\} \\ &= \frac{k^2}{2m}\left\{\frac{\phi(r)}{k - z_0} + r.t.\right\} \end{aligned} \tag{60}$$

The residue in the above equation is the following limit:

$$\lim_{k\mapsto z_0}(k - z_0)H\left\{\frac{\phi(r)}{k - z_0} + r.t.\right\} = \lim_{k\mapsto z_0}(k - z_0)\frac{k^2}{2m}\left\{\frac{\phi(r)}{k - z_0} + r.t.\right\}. \tag{61}$$

Equation (61) yields

$$H\phi(r) = \frac{z_0^2}{2m}\phi(r) \tag{62}$$

which shows that $\phi(r)$ is the eigenvector of H with eigenvalue $z_0^2/2m$. We apply this idea to (57), which has poles at γ_1 and γ_2. Then, the limits

$$\psi_{G_i} := \lim_{k\mapsto \gamma_i}(k - \gamma_i)\tilde{h}(k, r) \tag{63}$$

for $i = 1, 2$ give the Gamow vectors, which are:

$$\psi_{G_1} = \frac{(\gamma_1 - \beta_1)(\gamma_1 - \beta_2)}{(\gamma_1 - \gamma_2)}h(\gamma_1, r) \tag{64}$$

and

$$\psi_{G_2} = \frac{(\gamma_2 - \beta_1)(\gamma_2 - \beta_2)}{(\gamma_2 - \gamma_1)} h(\gamma_2, r) \tag{65}$$

As $R(k)$ does not depend on r and $h(k, r)$ satisfies the eigenvalue equation (58), so does $\tilde{h}(k, r)$ and therefore, ψ_{G_i}:

$$H\psi_{G_i} = \frac{\gamma_i^2}{2m}\psi_{G_i}, \quad i = 1, 2. \tag{66}$$

Thus, ψ_{G_i}, $i = 1, 2$, are the Gamow vectors in this case, as they are eigenvalues of the Hamiltonian with complex eigenvalues corresponding to the poles of $S_0(E)$.

From (65), (64), (52), and (46), we observe that

$$\lim_{r \mapsto \infty} \psi_{G_i}(r) = \infty, \quad i = 1, 2.$$

This is a result known since the first paper of Gamow (1928), i.e., that Gamow vectors grow at infinity in coordinate representation.

4. CONCLUDING REMARKS

We have studied two exactly solvable models in non relativistic Quantum Mechanics that exhibit resonance phenomena. The first model under our consideration is the Friedrichs model. We introduce the Friedrichs model in Hilbert space and in rigged Hilbert space languages and show the advantages of the latter. We present a detailed construction of Gamow vectors for single and double pole resonances in the Friedrichs model. The technique here used to construct the Gamow vectors can be applied to many other systems.

For spherically symmetric potentials, it is possible to place at selected points a finite number of resonances and obtain the S-matrix, for a fixed value of the angular momentum. The radial form of the potential producing these resonances is then known. This produces a second exactly solvable model having resonances in which the corresponding Gamow vectors can be obtained.

ACKNOWLEDGMENTS

We thank Profs. A. Bohm, I. Prigogine, and E. Hernández as well as Drs. R. de la Madrid and S. Wickramasekara for stimulating discussions and their interest in the present research. This work was financed by the Solvay Institutes, the Belgian Government through the Interuniversity Attraction Poles, the Junta de Castilla y León Project VA 085/02, and the Spanish DGI BMF 2002-3773 and DGI BMF 2002-0200.

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